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SHELLABILITY OF CHESSBOARD COMPLEXES

BY

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ABSTRACT

The matchings in a complete bipartite graph form a simplicial complex, which in many cases has strong structural properties. We use an equivalent description as **chessboard complexes**: the complexes of all non-taking rook positions on chessboards of various shapes.

In this paper we construct 'certificate k-shapes' $\Sigma(m,n,k)$ such that if the shape A contains some $\Sigma(m,n,k)$, then the $(k-1)$-skeleton of the chessboard complex $\Delta(A)$ is **vertex decomposable** in the sense of Provan & Billera. This covers, in particular, the case of rectangular chessboards $A=[m]\times[n]$, for which $\Delta(A)$ is vertex decomposable if $n \geq 2m-1$, and the $(\lfloor \frac{m+n+1}{3} \rfloor -1)$-skeleton is vertex decomposable in general.

The notion of vertex decomposability is a very convenient tool to prove shellability of such combinatorially defined simplicial complexes. We establish a relation between vertex decomposability and the CL-shellability technique (for posets) of Björner & Wachs.

0. Introduction

Consider a chessboard of size $m \times n$. [We will assume $m \leq n$ for this introduction.] Every **non-taking rook configuration** (that is, no two rooks on the same row or column) on the chessboard can be identified with the set of squares it occupies. The set of all such rook configurations forms an abstract simplicial complex: the empty set of rooks is non-taking, and any subset of a non-taking configuration is non-taking as well.

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This simplicial complex, the **chessboard complex** $\Delta_{m,n}$, appears in several interesting combinatorial situations: as a coset complex $K(m,n)$ of certain subgroups in the symmetric group studied by Garst [Ga, Chap. 3], as a multiple deleted join of a 0-complex as studied by Sarkaria [Sa], as a complex of injective functions $\mathcal{P}_{m,n}$ in the analysis of Tverberg-type problems by Vrećica & Živaljević [VZ, Sect. 2], as the matching complex $M(K_{m,n})$ of a complete bipartite graph (see Lovász & Plummer [LP]), thus as an intersection of two partition matroids, and probably in many more situations — see also the introduction of [BLVZ].

Since the chessboard complexes are so easy to define and appear to be the combinatorial essence in such diverse situations, there is a strong interest in understanding their combinatorial and topological properties. In particular, several of the applications (as in [VZ]) need information on connectivity properties.

Recall the following (well-known) hierarchy of properties of simplicial complexes:

$$\text{vertex decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{homotopy CM} \Longrightarrow \text{CM}.$$

The first result for chessboard complexes was Garst's Theorem [Ga, Thm. 15] in 1979: $\Delta_{m,n}$ is Cohen-Macaulay (CM) if and only if $n \geq 2m-1$. This was recently strengthened by Björner, Lovász, Vrećica & Živaljević [BLVZ]: the complexes are always $\min\{m-2, \lfloor\frac{n+m+1}{3}\rfloor-2\}$-connected, so the $(\lfloor\frac{n+m+1}{3}\rfloor-1)$-skeleton of $\Delta_{m,n}$ is homotopy Cohen-Macaulay. The Cohen-Macaulay property has strong enumerative consequences (see [Bj2, Sect. 7.5]): we get information on the matching polynomials of complete bipartite graphs [LP] from this.

On the combinatorial side this suggests — see the 'Final Remark' by Björner, Lovász, Vrećica & Živaljević — that $\Delta_{m,n}$ should be shellable for $n \geq 2m-1$, and that the $(\lfloor\frac{n+m+1}{3}\rfloor-1)$-skeleton of $\Delta_{m,n}$ should be shellable. [Whoever thought about this noticed, however, that the 'obvious approach' does not work: the lexicographic ordering on the facets does not produce a shelling for the natural orderings on the vertices; also, it is not clear at first how the condition $n \geq 2m-1$ should come into such a proof.] Shellability is a strong statement: in addition to homotopy Cohen-Macaulayness it implies (at least in principle) the construction of a distinguished homology basis, see Björner's discussion in [Bj2, Sect. 7.7].

Here we establish an even stronger condition: the $(\lfloor\frac{n+m+1}{3}\rfloor-1)$-skeleton of $\Delta_{m,n}$ is vertex decomposable in the sense of Provan & Billera [BP] [PB], see Section 1. According to Provan & Billera [PB, Thm. 2.10] this additionally implies the "Hirsch bound" on the diameter; applied to the skeleta of chessboard complexes $\Delta_{m,n}$ it says that every non-taking rook position of k rooks on the $(m\times n)$-chessboard can be transformed into any other position in at most $mn-k$ single-rook moves, so that every intermediate position is non-taking as well, if $k \leq \lfloor\frac{n+m+1}{3}\rfloor$. It seems that this upper bound for the number of moves is non-trivial, although by far not best possible. It would be interesting to determine the exact bound.

Our proofs of vertex decomposability will apply to the skeleta of chessboard complexes of quite arbitrary shapes (corresponding to matching complexes of general bipartite graphs). Following [BLVZ], we will identify the squares in the 'infinite chessboard' with $\mathbb{Z}^2$ in matrix notation, where the square (i,j) lies i rows below and j columns to the right of some reference point/square $(0,0)$.

In the course of the investigation it turns out that the structure that determines the shellability of the 'classical' rectangular chessboard complex is the largest **diamond shape**

$$\Sigma_m := \{(i,j) \in [m]\times\mathbb{Z}: 0 \leq j-i \leq m-1\}.$$

it contains, where we use the notation $[m] := \{1,\ldots,m\}$. Figure 0.1 shows the diamond shapes $\Sigma_1, \Sigma_2, \Sigma_3$, where every square is labeled by the corresponding pair in $\mathbb{Z}^2$.

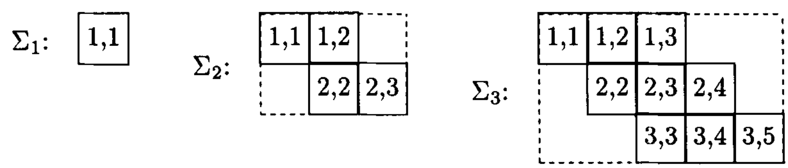


Figure 0.1: The diamond shapes Σ_m

With these shapes, we find that for any subset A of an $(m \times n)$-board that contains an isomorphic copy of the diamond board Σ_m (i.e., allowing for row and column permutations and transposition), the chessboard complex $\Delta(A)$ is shellable.

THEOREM 0.1: *Let $A \subseteq [m] \times \mathbb{Z}$ be a finite subset that contains the diamond shape Σ_m. Then $\Delta(A)$ is vertex decomposable of dimension $m-1$.*

This formulation is not only stronger than the statement for rectangular chessboards, it also admits a simple inductive proof. In this paper, we actually will give the proof in two versions. In Theorem 2.3 we will demonstrate the **simplicity** of the proof technique on the basic version for rectangular chessboards, which yields a statement that is weaker than Theorem 0.1. However, Theorem 0.1 can be proved along the same line, and in Theorem 3.3 we will show the **power** of the technique by proving the most general version we know for skeleta of complexes of planar shapes; this Theorem 3.3 also contains Theorem 0.1 as a very special case — see the end of Section 3.

1. Vertex decomposable complexes and shellability

An (abstract, finite) **simplicial complex** is a family of sets $\Delta \subseteq 2^E$ that contains $\emptyset$ and with any set also contains all its subsets. See Björner [Bj3] for a basic treatment of simplicial complexes and their combinatorics. Here we will only review the notions that are needed in the following.

We refer to E as the **ground set**, which contains the **set of vertices** $V(\Delta) = \{e \in E\colon \{e\} \in \Delta\}$. We admit the **empty complex** $\Delta = \{\emptyset\}$ in our discussions (and use it to start inductions on the size of the vertex set).

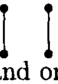
Using geometric language, we refer to the sets in Δ as **faces** of Δ, where the **dimension** of a face is one less than its cardinality: $\dim(A) = |A|-1$ for $A \in \Delta$. The **dimension** $\dim(\Delta)$ of Δ is the largest dimension of a face in Δ. A simplicial complex is **pure** if all its maximal faces have the same dimension. For example, a simplicial complex of dimension 0 is just a non-empty set of vertices. A simplicial complex of dimension 1 is a graph with at least one edge; it is pure if it does not have an isolated vertex.

A **simplex** is the simplicial complex given by all the subsets of a finite set. Thus every simplex is pure; the simplex of dimension -1 is the empty complex. Finally, the k-skeleton of Δ is the complex of all faces in Δ of dimension at most k:

$$\Delta^{\leq k} := \{A \in \Delta\colon |A| \leq k+1\}.$$

We will use the following notation for deletions, restrictions and links. If A is any subset of the ground set, then the **deletion** of A from Δ is $\Delta \setminus A := \{B \in \Delta\colon A \cap B = \emptyset\}$. In particular, we need the case where $A = \{v\}$ is a single vertex of Δ. In this case we write $\Delta \setminus v := \Delta \setminus \{v\}$. We will also use the **restriction** to a subset of the ground set: $\Delta(A) := \{B \in \Delta\colon B \subseteq A\} = \Delta \setminus (E \setminus A)$. Similarly, for any face $A \in \Delta$ the **link** is $\Delta/A := \{B \in \Delta\colon A \cap B = \emptyset, A \cup B \in \Delta\}$. Again, we write $\Delta/v := \Delta/\{v\}$ for a vertex v of Δ. [This notation follows matroid theory usage. One key observation is that deletions and links commute. Any deletion of a link, or a link of a deletion, will be referred to as a **minor** of the complex in question.] Using deletion and link of a vertex as primitives, we get the following recursive definition.

Definition 1.1: Provan & Billera [BP] [PB, Def. 2.1]. A simplicial complex Δ is **vertex decomposable** if it is pure and it is either empty, or it has a vertex v such that $\Delta\setminus v$ and Δ/v are vertex decomposable (of smaller size). ∎

(If Δ is pure of dimension k, then Δ/v is automatically pure of dimension $k-1$. If $\Delta\setminus v$ is also pure, then either $\dim(\Delta\setminus v) = k$, or we get that Δ is a cone over $\Delta\setminus v$, where $\Delta\setminus v = \Delta/v$ has dimension $k-1$. A good example of a complex that is pure but not vertex decomposable is the 1-dimensional complex  .)

Equivalently, a non-empty complex Δ is vertex decomposable if and only if it is pure and it has an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices such that

$\Delta\setminus\{v_i, \ldots, v_n\}$ and $\Delta/v_i\setminus\{v_{i+1}, \ldots, v_n\}$ are both vertex decomposable, for $1 \leq i \leq n$.

This is the criterion used in the proofs of this paper. In fact, using the induction hidden in this, it suffices to require

$\Delta\setminus\{v_i, \ldots, v_n\}$ is pure and $\Delta/v_i\setminus\{v_{i+1}, \ldots, v_n\}$ is vertex decomposable, for $1 \leq i \leq n$.

Thus the proof of vertex decomposability amounts to specifying a good vertex ordering for the complex (called **shedding orders** in [PB, p. 587]) and recursively for some of its minors.

For example, to show that $\Delta_{4,8}$ is vertex decomposable, our proof in Theorem 2.3 below shows that we can take any vertex ordering that picks the squares, for example, with increasing labels according to the following figure:

4	5	7	7	9	9	11	11
6	6	3	7	9	9	11	11
8	8	8	8	2	9	11	11
10	10	10	10	10	10	1	11

Figure 1.1: Shedding order for $\Delta_{4,8} = \Delta([4] \times [8])$

The following simple lemma will be the key to our inductive treatment of skeleta of complexes. Note that it contains the fact that the cone of a vertex decomposable complex is again vertex decomposable as a special case. The analogous statements for shellable and for Cohen-Macaulay complexes are also quite obvious.

LEMMA 1.2: *If Δ is a finite simplicial complex whose k-skeleton $\Delta^{\leq k}$ is vertex decomposable, then the $(k+1)$-skeleton $(\Delta * v)^{\leq k+1}$ of the cone over Δ is vertex decomposable as well.*

Proof: In fact, we show that if $(v_1, \ldots, v_t)$ is a shedding order for $\Delta^{\leq k}$, then $(v, v_1, \ldots, v_t)$ is a shedding order for $(\Delta * v)^{\leq k+1}$.

First we note that $(\Delta * v)^{\leq k+1}$ is a pure complex of dimension $k+1$, whose maximal faces are the k-faces of Δ augmented by v, and the $(k+1)$-faces of Δ. Now to see vertex decomposability, we use induction on t and simply compute

$(\Delta * v)^{\leq k+1}/v_t = ((\Delta * v)/v_t)^{\leq k} = ((\Delta/v_t) * v)^{\leq k}$, which is vertex decomposable by induction, and

$(\Delta * v)^{\leq k+1} \setminus v_t = ((\Delta * v) \setminus v_t)^{\leq k+1} = ((\Delta \setminus v_t) * v)^{\leq k+1}$, which is also vertex decomposable by induction. ∎

It seems to be a natural problem to relate vertex decomposability for complexes to the lexicographic shellability technique of Björner & Wachs [BjW].

PROPOSITION 1.3: *If Δ is vertex decomposable, then $P(\Delta)$ has a recursive atom ordering in the sense of* [BjW]. *The converse is false in general.*

Proof: We prove, more precisely, that any shedding order for Δ is a recursive atom ordering for the face poset $P(\Delta)$. Since $P(\Delta)$ is a semilattice, we can use the Wachs & Walker formulation of recursive atom orderings [WW, Sect. 7]: we have to prove that if $(v_1, \ldots, v_t)$ is a shedding order for Δ, then Δ/v_i has a shedding order in which the vertices in $\{v_j : j<i, \{v_i, v_j\} \in \Delta\}$ come first.

Thus it suffices to verify the following claim: if $(v_1, \ldots, v_t)$ is a shedding order for Δ, and $(v'_1, \ldots, v'_{i'})$ is a shedding order for $\Delta/v_i \setminus \{v_{i+1}, \ldots, v_t\}$, then $(v'_1, \ldots, v'_{i'}, v_{i+1}, \ldots, v_t)$ is a shedding order for Δ/v_i. (From these shedding orders one could delete the elements v_j $(j > i)$ such that $\{v_i, v_j\} \notin \Delta$, for which v_j is not a vertex of the complex Δ/v_i.)

This is easily proved by induction on $t-i$, using some observations in the proof of [PB, Prop. 2.3]. The case $\{v_i, v_t\} \notin \Delta$ corresponds to deleting an irrelevant element which is not in the ground set. Now if $\{v_i, v_t\} \in \Delta$, then we compute $(\Delta/v_i)\setminus v_t = (\Delta\setminus v_t)/v_i$ and $(\Delta/v_i)/v_t = (\Delta/v_t)/v_i$. In both cases we are done by induction.

For the converse, consider the boundary complexes of simplicial polytopes. These complexes are not all vertex decomposable [KK, Sects. 6.3, 6.4], although their face posets have recursive atom orderings (RAO) by [BjW, Thm. 4.5]. ∎

Altogether we think that one should have the hierarchy:

$$\Delta \text{ vertex decomposable} \Longrightarrow P(\Delta) \text{ has RAO} \Longrightarrow \Delta \text{ shellable}$$

Here Proposition 1.3 proves the first implication and shows that the converse is false. The second implication remains a conjecture, where we do not know about the converse either. However, it is clear that vertex decomposability of Δ implies shellability, by Provan & Billera [BP] [PB, Cor. 2.9]. We note that the chain of implications can be continued as

$$\Delta \text{ shellable} \Longleftrightarrow P(\Delta) \text{ has RCO} \Longrightarrow \Delta(P(\Delta)) \text{ vertex decomposable}$$

The first equivalence, between shellability and recursive coatom orderings (RCO), is a main result of [BjW], while the second implication, involving the barycentric subdivision $\mathrm{sd}(\Delta) = \Delta(P(\Delta))$ of Δ, is an unpublished result of A. Björner [Bj4].

2. Rectangular chessboard complexes

Definition 2.1: For any (finite) subset $A \subseteq \mathbb{Z}^2$, we define the **generalized chessboard complex of** A as the simplicial complex

$$\Delta(A) := \{B \subseteq A\colon i{\neq}i' \text{ and } j{\neq}j' \text{ for } (i,j),(i',j'){\in}B,\ (i,j){\neq}(i',j')\}. \quad ∎$$

For this, we can view $\mathbb{Z}^2$ as the ground set, and let

$$\Delta(\mathbb{Z}^2) = \{B{\subseteq}\mathbb{Z}^2\colon i{\neq}i' \text{ and } j{\neq}j' \text{ for } (i,j),(i',j') \in B,\ (i,j){\neq}(i',j')\}$$

be the chessboard complex on the complete infinite chessboard, so that for a finite set $A \subseteq \mathbb{Z}^2$, the chessboard complex $\Delta(A)$ is the restriction of the infinite, infinite-dimensional complex $\Delta(\mathbb{Z}^2)$ to A.

In particular, this definition includes the 'classical' rectangular chessboard complexes as $\Delta_{m,n} = \Delta\big([m]\times[n]\big)$.

LEMMA 2.2: *Let A be a finite subset of $\mathbb{Z}^2$, and $(i,j) \in A$. Then the deletion and the link of the vertex (i,j) in the chessboard complex $\Delta(A)$ are given by*

$$\Delta(A)\setminus(i,j) = \Delta(A\setminus(i,j)) \quad \text{and} \quad \Delta(A)/(i,j) \cong \Delta(A_{i,j}),$$

where $A_{i,j}$ is the set of all $(i',j') \in \mathbb{Z}^2$ such that we have $(i'',j'') \in A$ with $i' = i'' < i$ or $i'+1 = i'' > i$, and with $j' = j'' < j$ or $j'+1 = j'' > j$.

Proof: This just states that the deletion of a vertex from a chessboard complex corresponds to deleting the corresponding square from the board, whereas the link is obtained by removing the corresponding row and column from the board, where we get an isomorphic complex by "closing the gap". ∎

THEOREM 2.3: *If A is a finite set with $[m]\times[2m-1] \subseteq A \subseteq [m]\times\mathbb{Z}$, then $\Delta(A)$ is a vertex decomposable complex of dimension $m-1$.*

Proof: We proceed by induction on m, the case $m = 1$ being trivial. For $m > 1$, we use that by induction

$$\Delta_{m-1,2m-2} = \Delta([m-1]\times[2m-2])$$

is vertex decomposable of dimension $m-2$. Now the complex

$$\Delta([m-1]\times[2m-2] \cup \{(m, 2m-1)\})$$

is a cone over $\Delta_{m-1,2m-2}$ and thus vertex decomposable of dimension $m-1$.

From this we use induction on $|A|$ to see that $\Delta(A)$ is vertex decomposable of dimension $m-1$ for

$$[m-1]\times[2m-2] \cup \{(m,2m-1)\} \subseteq A \subseteq [m]\times[2m-2] \cup \{(m,2m-1)\}.$$

In fact, the links we have to consider in the induction steps are isomorphic to $\Delta(A_{m,i})$ with $1 \leq i \leq 2m-2$ with $A_{m,i} = [m-1]\times[2m-3]$, so by induction they are vertex decomposable of dimension $m-2$.

To complete the argument, we show that whenever A is a finite set with

$$[m]\times[2m-2] \cup \{(m,2m-1)\} \subseteq A \subseteq [m]\times\mathbb{Z},$$

then $\Delta(A)$ is vertex decomposable of dimension $m-1$. Again we use induction on $|A|$, where we already know the claim for the case

$$[m]\times[2m-2] \cup \{(m,2m-1)\} = A.$$

Now if $(i,j) \in A$ with $j \in \mathbb{Z} \setminus [2m-2]$, then $[m-1] \times [2m-3] \subseteq A_{i,j}$, and thus $\Delta(A_{i,j})$ is vertex decomposable of dimension $m-2$ by induction. ∎

3. Skeleta of chessboard complexes

Definition 3.1: A **k-shape** is a subset $\Sigma(m,n,k) \subseteq \mathbb{Z}^2$ given by

$$\Sigma(m,n,k) := \{(i,j) \in [m]\times[n]: -(2k-1)+n \leq j-i \leq (2k-1)-m\}.$$

A k-shape is **admissible** if $m,n \leq 2k-1$ and $m+n+1 \geq 3k$.

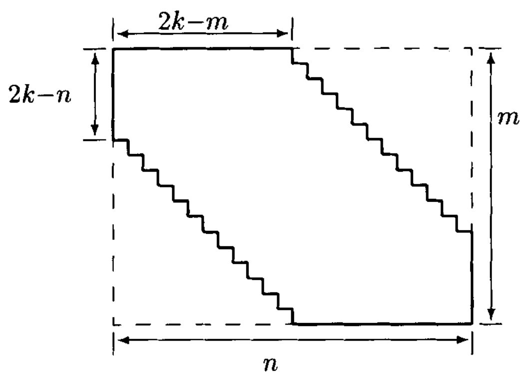


Figure 3.1: A typical admissible shape, $\Sigma(18,24,15)$

For our induction we use that from a k-shape A, the following operations still leave a shape that contains (an isomorphic copy of) an admissible $(k-1)$-shape:

(a) deleting the row and column of a square outside A,
(b) deleting the last row and the last column,
(c) deleting the last row and the last column, plus another row (column) that contains a square in the last column (row).

LEMMA 3.2: *For $S(a,b) := \{(i,j) \in \mathbb{Z}^2: a \leq j-i \leq b\}$ with $a < b$ and $(i,j) \in \mathbb{Z}^2$ we have*

$$S(a,b)_{i,j} \supset S(a+1,b) \quad \text{for } j-i > b,$$
$$S(a,b)_{i,j} \supset S(a,b-1) \quad \text{for } j-i < a.$$

Proof: There are three simple cases to check for the first claim (a sketch will help). The second one follows by symmetry. ∎

With the notation of Lemma 3.2, we have

$$\Sigma(m,n,k) \;=\; [m]\times[n] \;\cap\; S(-2k+1+n, 2k-1-m).$$

THEOREM 3.3: *If $A \subseteq \mathbb{Z}^2$ is a finite set that contains (an isomorphic copy of) an admissible k-shape, then $\Delta(A)^{\leq k-1}$ is vertex decomposable of dimension $k-1$.*

Proof: We proceed by induction on k, the case $k = 1$ being trivial: the only admissible 1-shape is $\{(1,1)\}$. For $k > 1$ note that

$$\Sigma(m,n,k)_{m,n} \;=\; [m-1]\times[n-1] \;\cap\; S(-2k+1+n, 2k-1-m)$$

contains an admissible $(k-1)$-shape, namely

$$\begin{aligned} &\Sigma(m-2, n-1, k-1) && \text{if } n < 2k-1, \\ &\Sigma(m-1, n-2, k-1) && \text{if } m < 2k-1, \text{ and} \\ &\Sigma(m-2, n-2, k-1) && \text{if } m = n = 2k-1. \end{aligned}$$

From this, and Lemma 1.2, we get that the skeleton $\Delta(A)^{\leq k-1}$ of the chessboard complex for $A = \Sigma(m,n,k)_{m,n} \cup \{(m,n)\}$ is vertex decomposable of dimension $k-1$.

As the next step we show that adding the squares of the last row and column to $A = \Sigma(m,n,k)_{m,n} \cup \{(m,n)\}$ preserves vertex decomposability of the $(k-1)$-skeleton. That is, we claim that $\Delta(A')$ is vertex decomposable for

$$\Sigma(m,n,k)_{m,n} \cup \{(m,n)\} \;\subseteq\; A' \;\subseteq\; \Sigma(m,n,k).$$

Using induction on $|A'|$ and the symmetry between m and n, we show that $A'_{i,n}$ contains an admissible $(k-1)$-shape for $n-(2k-1)+m \leq i < m$. With this we have $2k-1 < n$, and using Lemma 3.2 we compute

$$\begin{aligned} A'_{i,n} &= \big([m-1]\times[n-1]\big)_{i,n} \;\cap\; \big(S(-(2k-1)+n, (2k-1)-m)\big)_{i,m+n} \\ &\supseteq [m-2]\times[n-1] \;\cap\; S(-(2k-1)+n+1, (2k-1)-m) \\ &= \Sigma(m-2, n-1, k-1). \end{aligned}$$

Symmetrically, for any j with $m-(2k-1)+n \leq j < n$ the minor $A'_{m,j}$ contains an admissible $(k-1)$-shape, namely $\Sigma(m-1, n-2, k-1)$. With this we know that $\Delta\big(\Sigma(m,n,k)\big)^{\leq k-1}$ is vertex decomposable of dimension $k-1$.

Now let $\Sigma(m,n,k) \subseteq A$; we use induction on $|A|$ to show that $\Delta(A)^{\leq k-1}$ is vertex decomposable of dimension $k-1$. For this, we prove that if $(i,j) \in$

$\mathbb{Z}^2 \setminus \Sigma(m,n,k)$, then $\Sigma(m,n,k)_{i,j}$ contains an admissible $(k-1)$-shape, or a translate of one.

First assume that $(i,j) \in [m]\times[n]$. By symmetry, we may assume $j-i > 2k-1-m$, and compute

$$\begin{aligned}
\Sigma(m,n,k)_{i,j} &= [m-1]\times[n-1] \cap S(-2k+1+n, 2k-1-m)_{i,j} \\
&\supseteq [m-1]\times[n-1] \cap S(-2k+2+n, 2k-1-m)
\end{aligned}$$

$$\begin{cases}
\supseteq [m-2]\times[n-1] \cap S(-2k+2+n, 2k-1-m) \\
\quad = \Sigma(m-2, n-1, k-1), \\
\quad \text{which is admissible if } n < 2k-1, \\
\supseteq [m-2]\times[n-2] \cap S(-2k+2+n, 2k-2-m) \\
\quad = \Sigma(m-2, n-2, k-1), \\
\quad \text{which is admissible if } m+n+1 > 3k, \\
\supseteq [m-2]\times[n-2] + \binom{0}{1} \cap S(-2k+1+n, 2k-2-m) + \binom{0}{1} \\
\quad = \Sigma(m-1, n-2, k-1) + \binom{0}{1}, \\
\quad \text{which is admissible if } m < 2k-1.
\end{cases}$$

where one of the three last cases applies: otherwise we would have $n = 2k-1$, $m+n+1 = 3k$ and $m = 2k-1$, thus $k = 1$.

Now we treat the case where $i \in [m]$, but $j \in \mathbb{Z}\setminus[m]$. This corresponds to deleting one arbitrary row from $\Sigma(m,n,k)$. Since the column corresponding to j does not hit $\Sigma(m,n,k)$, we may assume that j is large, $j-i > n$. Thus we get

$$\begin{aligned}
\Sigma(m,n,k)_{i,j} &= [m-1]\times[n] \cap S(-2k+1+n, 2k-1-m)_{i,j} \\
&\supseteq [m-1]\times[n] \cap S(-2k+2+n, 2k-1-m).
\end{aligned}$$

This contains

$$\begin{aligned}
&\Sigma(m-1, n-2, k-1), && \text{which is admissible if } m < 2k-1, \\
&\Sigma(m-2, n-1, k-1), && \text{which is admissible if } n < 2k-1, \\
&\Sigma(m-2, n-2, k-1), && \text{which is admissible if } m+n+1 > 3k
\end{aligned}$$

and one of the three cases occurs. Now again the case of $j \in [n]$, $i \in \mathbb{Z}\setminus[m]$ follows by symmetry, and the last case, where $i \notin [m]$, $j \notin [n]$, is implied by any of the three previous ones. ∎

In particular, let $A = [m]\times[n]$ be a rectangular shape, and assume $m \leq n$ (without loss of generality). The complex $\Delta(A)$ has dimension $m-1$. If $n \geq 2m-1$, then $\Sigma_m = \Sigma(m, 2m-1, m) \subseteq A$, and thus by Theorem 3.3 $\Delta(A)^{\leq m-1} = \Delta(A)$ is vertex decomposable. (This proves Theorem 0.1.) If $m \leq n \leq 2m-1$, then for $k := \lfloor \frac{m+n+1}{3} \rfloor$ we get that $\Sigma(m,n,k)$ is admissible, and thus the $(k-1)$-skeleton $\Delta(A)$ is vertex decomposable. The conjecture of Björner, Lovász, Vrećica & Živaljević [BLVZ, Conj. 1.5(a)] would imply that this $k = \lfloor \frac{m+n+1}{3} \rfloor$ is maximal.

As a special case we meet the "challenge" of Björner, Lovász, Vrećica & Živaljević [BLVZ, Sect. 5]: the complex of non-taking positions of at most 5 rooks on an (8×8)-chessboard is shellable: and our description implicitly contains an explicit shelling of $\Delta([8]\times[8])^{\leq 4}$. In fact, the same is true for the 4-skeleton of the (7×7)-board, because $\Sigma(7,7,5) \subseteq [7]\times[7] \subseteq [8]\times[8]$ is admissible.

4. Final remarks

The following characterization of matroid complexes is well-known (see [PB, Prop. 3.2.3] [Bj2, Ex. 7.4, Thm. 7.3.4]): if every restriction of Δ to a subset (including the complex Δ itself) is pure, then every such restriction is a matroid complex, and thus in particular vertex decomposable.

In the following sense the chessboard complexes are very 'close' to being matroid complexes. We propose to call (Δ, A) a **relative matroid complex** if Δ is a simplicial complex, A is a finite subset of its vertex set, and every finite minor of Δ that contains A (i.e., obtained by deleting or taking links of elements not in A) is vertex decomposable. Equivalently, $\Delta(A)$ is vertex decomposable, and every finite minor of Δ that contains A is pure. With this, the case $A = \emptyset$ corresponds to a usual matroid complex, whereas Theorem 3.3 establishes that $(\Delta(\mathbb{Z}^2)^{\leq k-1}, \Sigma)$ is a relative matroid complex whenever Σ contains an admissible k-shape of Definition 3.1. It might be interesting to study the exchange properties of such relative matroid complexes, with the aim of deriving diameter bounds that improve upon the Hirsch bounds (cf. the introduction).

In view of Theorem 3.3 one could ask for a complete list of minimal certificates for vertex decomposability, that is, all minimal shapes (up to isomorphism) that determine a relative matroid complex. Here we only note that the list given by Theorem 3.3 is not itself complete: for example, $A = \{(1,1),(1,2),(2,3),(2,4)\}$ is a minimal certificate shape for $k = 2$, distinct from the admissible shapes $\Sigma(2,3,2) = \Sigma_2$, its transpose $\Sigma(3,2,2)$, and $\Sigma(3,3,2) = \{(1,1),(2,2),(3,3)\}$, as given by Definition 3.1.

However, the skeletal dimension $k-1$ is best possible for each admissible shape $\Sigma(m,n,k)$. In fact, consider the set

$$A := \{(i+m-k, i) \colon 1 \leq i \leq k\} \cup \Sigma(m,n,k).$$

Then $\{(i+m-k, i): 1 \le i \le k\}$ is a maximal face of $\Delta(A)$, so the k-skeleton $\Delta(A)^{\le k}$ cannot be pure of dimension k.

To treat infinite complexes, we need to adapt the notion of vertex decomposability; this is quite straightforward for the definitions given after Definition 1.1. (One can rely on Björner's treatment of infinite shellable complexes [Bj1, Sect. 1(A)] for guidance.) With this, we can drop the finiteness assumption on A in Theorem 3.3. In particular, we get connectivity results for $\Delta(A)$ also if A is infinite (using compactness arguments). It is easy to see that $\Delta(\mathbb{Z}^2)$ is in fact contractible.

It seems likely that our method can be applied to treat the complexes of higher-dimensional chessboards as well, which corresponds to the matching complexes of balanced complete hypergraphs. The connectivity results of Björner, Lovász, Vrećica & Živaljević extend to this setting, see [BLVZ, Sect. 4]. They were further generalized by H. Eriksson [Bj4]. We will not pursue this here.

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References

[BP] L. J. Billera and J. S. Provan, *A decomposition property for simplicial complexes and its relation to diameters and shellings*, Ann. NY Acad. Sci. **319** (1979), 82–85.

[Bj1] A. Björner, *Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings*, Advances in Math. **52** (1984), 173–212.

[Bj2] A. Björner, *Homology and shellability of matroids and geometric lattices*, in *Matroid Applications* (N. White, ed.), Cambridge University Press, 1992, pp. 226–283.

[Bj3] A. Björner, *Topological Methods*, in *Handbook of Combinatorics* (R. Graham, M. Grötschel and L. Lovász, eds.), North-Holland, Amsterdam, to appear.

[Bj4] A. Björner, personal communication.

[BLVZ] A. Björner, L. Lovász, S. T. Vrećica and R. T. Živaljević, *Chessboard complexes and matching complexes*, J. London Math. Soc., to appear.

[BjW] A. Björner and M. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), 323–341.

[Ga] P. F. Garst, *Cohen-Macaulay Complexes and Group Actions*, Ph.D. Thesis, University of Wisconsin-Madison, 1979, 130 pp.

[KK] V. Klee and P. Kleinschmidt, *The d-step conjecture and its relatives*, Math. Operations Research **12** (1987), 718–755.

[LP] L. Lovász and M. D. Plummer, *Matching Theory*, Akadémiai Kiadó, Budapest, and North-Holland, Amsterdam, 1986.

[PB] J. S. Provan and L. J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Operations Research **5** (1980), 576–594.

[Sa] K. S. Sarkaria, *A generalized van Kampen–Flores theorem*, Proc. Amer. Math. Soc. **111** (1991), 559–565.

[VZ] S. T. Vrećica and R. T. Živaljević, *The colored Tverberg's problem and complexes of injective functions*, J. Combinatorial Theory, Ser. A **61** (1992), 309–318.

[WW] M. L. Wachs and J. W. Walker, *On geometric semilattices*, Order **2** (1986), 367–385.